

THE TUTTE POLYNOMIAL OF THE SCHREIER GRAPHS OF THE GRIGORCHUK GROUP AND THE BASILICA GROUP

TULLIO CECCHERINI-SILBERSTEIN, ALFREDO DONNO, AND DONATELLA IACONO

ABSTRACT. We study the Tutte polynomial of two infinite families of finite graphs. These are the Schreier graphs associated with the action of two well-known self-similar groups acting on the binary rooted tree by automorphisms: the first Grigorchuk group of intermediate growth, and the iterated monodromy group of the complex polynomial $z^2 - 1$ known as the Basilica group. For both of them, we describe the Tutte polynomial and we compute several special evaluations of it, giving further information about the combinatorial structure of these graphs.

1. INTRODUCTION

The Tutte polynomial is a two-variable polynomial which can be associated with a graph, a matrix, or, more generally, with a matroid. It has many interesting applications in several areas of sciences as, for instance, Combinatorics, Probability, Statistical Mechanics, Computer Science and Biology. It was introduced by W.T. Tutte [18, 19, 20] and we will mainly refer to [4, 5, 11, 21] as expository papers.

Given a finite graph G , its Tutte polynomial $T(G; x, y)$ satisfies a fundamental universal property with respect to the deletion-contraction reduction of the graph. Hence, any multiplicative graph invariant with respect to a deletion-contraction reduction turns out to be an evaluation of it. This polynomial is quite interesting since several combinatorial, enumerative and algebraic properties of the graph – such as the number of spanning trees, of spanning connected subgraphs, of spanning forests and of acyclic orientations of the graph – can be investigated by considering special evaluations of this polynomial. Moreover, from the Tutte polynomial one also recovers the reliability and the chromatic polynomials. It has also many interesting connections with statistical mechanical models as the Potts model [22], percolation [17], the Abelian Sandpile Model [7, 15], as well as with the theory of error correcting codes [22].

In this paper, we study the Tutte polynomial of the Schreier graphs associated with the action of two well-known automorphism groups of the binary rooted tree: the Grigorchuk group and the iterated monodromy group of the complex polynomial $z^2 - 1$ known as the Basilica group. See also [10], where the Tutte polynomials of the Sierpiński graphs and the Schreier graphs of the Hanoi Towers group $H^{(3)}$ are computed.

The first Grigorchuk was introduced by R. Grigorchuk in 1980; it yields the simplest solution of the Burnside problem (an infinite, finitely generated torsion group) and the first example of a finitely generated group of intermediate (i.e. faster than polynomial but slower than exponential) growth. See [6] and [12] for a detailed account and further references.

Mathematics Subject Classification (2010): 05C31, 05C15, 05C30, 05C38, 20E08.

Date: July 17, 2010.

Key words and phrases. Tutte polynomial, Schreier graph, Grigorchuk group, Basilica group, spanning subgraph, acyclic orientation, reliability polynomial, chromatic polynomial, partition function of the Ising model.

The Basilica group was introduced by R. Grigorchuk and A. Żuk in [13] as a group generated by a three-state automaton. It is a remarkable fact due to V. Nekrashevych [16] that this group can be described as the iterated monodromy group of the complex polynomial $z^2 - 1$; therefore, there exists a natural way to associate with it a compact limit space homeomorphic to the well-known Basilica fractal. Moreover, it is the first example of an amenable group (a highly non-trivial and deep result of L. Bartholdi and B. Virág [3]) not belonging to the class of subexponentially amenable groups, which is the smallest class containing all groups of subexponential growth and closed after taking subgroups, quotients, extensions and direct unions.

Over the last decade, Grigorchuk and a number of coauthors have developed a new exciting direction of research focusing on finitely generated groups acting by automorphisms on rooted trees, transitively on each level [2]. They proved that these groups have deep connections with the theory of profinite groups and with complex dynamics. In particular, many groups of this type satisfy a property of self-similarity (see Definition 2.8), reflected on fractalness of some limit objects associated with them [16].

In Sections 3 and 4, we study the Tutte polynomial for the Schreier graphs $\{\Gamma_n\}_{n \geq 1}$ and $\{B_n\}_{n \geq 1}$ of the Grigorchuk group and the Basilica group, respectively. It follows from the recursive expression of the generators of these groups that these graphs have a cactus structure, i.e., they are union of cycles, arranged in a tree-like way. This enables us to compute the Tutte polynomial using the multiplicative property (2) (see Section 2.1). Once we have these polynomials, we compute many special evaluations of them, providing several interesting information about the combinatorial structure of these graphs and showing connections with reliability, colorability and the Ising model (see Section 2.1 for definitions and details). Note that some evaluations of the Tutte polynomial are trivial, when the graphs $\{\Gamma_n\}_{n \geq 1}$ and $\{B_n\}_{n \geq 1}$ are considered with loops; for instance, the number of acyclic orientations, the chromatic polynomial and the partition function of the Ising model. Therefore, we make these computations on the graphs $\{\Gamma_n^*\}_{n \geq 1}$ and $\{B_n^*\}_{n \geq 1}$, obtained from the graphs $\{\Gamma_n\}_{n \geq 1}$ and $\{B_n\}_{n \geq 1}$, respectively, by deleting loops. For the considered graphs, we explicitly describe:

- *the Tutte polynomial* (Theorems 3.1 and 4.2);
- *the reliability polynomial* (Propositions 3.3 and 4.4);
- *the complexity* (Propositions 3.5 and 4.6);
- *the number of connected spanning subgraphs* (Propositions 3.8 and 4.9);
- *the number of spanning forests* (Propositions 3.10 and 4.11);
- *the number of acyclic orientations* (Propositions 3.13 and 4.14);
- *the chromatic polynomial* (Propositions 3.15 and 4.16);
- *the partition function of the Ising model* (Theorems 3.17 and 4.18).

2. PRELIMINARIES

2.1. The Tutte polynomial. Throughout the paper, we deal with graphs which are connected and finite. Moreover, both multiple edges and multiple loops are allowed. As usual, $G = (V(G), E(G))$ denotes a graph with vertex set $V(G)$ and edge set $E(G)$; we will often write V and E , when there is no risk of confusion, and so $G = (V, E)$. Moreover, we denote by E_n the graph with n vertices and no edges, and by K_n the complete graph on n vertices. A subgraph $A = (V(A), E(A))$ of a graph $G = (V(G), E(G))$ is said to be *spanning* if the condition $V(A) = V(G)$ is satisfied. In particular, a *spanning subtree* of G is a spanning subgraph of G

which is a tree, a *spanning forest* of G is a spanning subgraph of G which is a forest. The number of spanning trees of a graph G is called *complexity* of G and is denoted by $\tau(G)$. It is interesting to study complexity when the system grows. More precisely, given a sequence $\{G_n\}_{n \geq 1}$ of finite graphs with complexity $\tau(G_n)$, such that $|V(G_n)| \rightarrow \infty$, the limit

$$\lim_{|V(G_n)| \rightarrow \infty} \frac{\log \tau(G_n)}{|V(G_n)|},$$

when it exists, is called the *asymptotic growth constant* of the spanning trees of $\{G_n\}_{n \geq 1}$ [14]. Finally, let $k(G)$ be the number of connected components of G .

Definition 2.1. Let A be a spanning subgraph of G , then the rank $r(A)$ and the nullity $n(A)$ of A are defined as

$$r(A) = |V(A)| - k(A) = |V(G)| - k(A) \quad \text{and} \quad n(A) = |E(A)| - r(A) = |E(A)| - |V(A)| + k(A).$$

Definition 2.2 (Spanning subgraphs). Let $G = (V, E)$ be a graph. The Tutte polynomial $T(G; x, y)$ of G is defined as

$$(1) \quad T(G; x, y) = \sum_{A \subseteq G} (x-1)^{r(G)-r(A)} (y-1)^{n(A)},$$

where the sum runs over all the spanning subgraphs A of G .

The Tutte polynomial can be also defined by a recursion process given by deleting and contracting edges. We recall that, given $G = (V, E)$, the graph $G \setminus e = (V, E - \{e\})$ is obtained from G by deleting the edge $e \in E$. The graph obtained by contracting an edge $e \in E$ is the result of the identification of the endpoints of e followed by removing e . We denote it by G/e . Finally, we recall that an edge in a connected graph is a *bridge* if its deletion disconnects the graph, it is a *loop* if its endpoints coincide.

Definition 2.3 (Deletion-Contraction). Let $G = (V, E)$ be a graph. The Tutte polynomial $T(G; x, y)$ of G is defined as

$$T(G; x, y) = \begin{cases} 1 & \text{if } G = E_1; \\ xT(G \setminus e; x, y) & \text{if } e \text{ is a bridge;} \\ yT(G \setminus e; x, y) & \text{if } e \text{ is a loop;} \\ T(G \setminus e; x, y) + T(G/e; x, y) & \text{if } e \text{ is neither a bridge nor a loop.} \end{cases}$$

The recursive process to compute the Tutte polynomial in this second definition is independent on the order in which the edges are chosen: this can be proven by showing that Definitions 2.2 and 2.3 are equivalent [5].

Once we have the definition, we can state some of the main properties of the Tutte polynomial (for more details, see [4, 5, 11]). Recall that a one point join $G * H$ of two graphs G and H is obtained by identifying a vertex v of G and a vertex w of H into a single vertex of $G * H$. The following property can be easily proven by using Definition 2.2:

$$(2) \quad T(G * H; x, y) = T(G; x, y)T(H; x, y).$$

This property will be fundamental for our computations and we will refer to it as the *multiplicative property* of the Tutte polynomial. The next lemma follows from Definition 2.3, by using the multiplicative property.

Lemma 2.4. If C_k is a cycle of length k , with $k \geq 2$, then its Tutte polynomial $T(C_k; x, y)$ is $y + x + x^2 + \dots + x^{k-1}$.

Proof. The proof is by induction on the length k of the cycle. For $k = 2$, C_2 is a 2-cycle. It is trivial to verify that the Tutte polynomial of the graph



is x , using Definition 2.3. Therefore, for the graph C_2



the Tutte polynomial is $T(C_2; x, y) = y + x$, again by Definition 2.3, so that the assertion is true for $k = 2$. Now let C_{k+1} be a cycle of length $k + 1$. Let e be a fixed edge of C_{k+1} , so that by Definition 2.3,

$$T(C_{k+1}; x, y) = T(C_k; x, y) + T(C_{k+1} \setminus e; x, y).$$

Since, by the multiplicative property, $T(C_{k+1} \setminus e; x, y) = x^k$, we can apply induction and we get the required result. \square

In the next sections, we will be interested in special evaluations of the Tutte polynomial, that allow us to deduce many combinatorial and algebraic properties of the graphs considered. In the following theorem, we collect many of these properties that are well-known in literature.

Theorem 2.5. [11, Theorem 3 and 8] *Let $G = (V, E)$ be a connected graph and denote by $T(G; x, y)$ its Tutte polynomial. Then:*

- (1) $T(G; 1, 1) = \tau(G)$;
- (2) $T(G; 1, 2)$ is the number of spanning connected subgraphs of G ;
- (3) $T(G; 2, 1)$ is the number of spanning forests of G ;
- (4) $T(G; 2, 2) = 2^{|E|}$;
- (5) $T(G; 2, 0)$ is the number of acyclic orientations of G , i.e., orientations having no oriented cycles.

Another fundamental interesting aspect of the Tutte polynomial is that, starting from it, one can obtain other interesting polynomials associated with the graph: the *reliability polynomial* and the *chromatic polynomial*.

More precisely, as regards the reliability polynomial $R(G, p)$, suppose that each edge of G is independently chosen to be active (or open) with probability p or inactive (closed) with probability $1 - p$. Then, $R(G, p)$ is defined as the probability that in this random model there is a path of active edges between each pair of vertices of G .

As regards the colorability, we recall that a proper (or admissible) λ -coloring of the vertices of G is an assignment of λ colors to the vertices of G , in such a way that adjacent vertices have distinct colors. G is said λ -colorable if it admits a proper λ -coloring. The *chromatic number* $\chi(G)$ of G is defined as the minimal number λ such that G is λ -colorable. G is uniquely λ -colorable if $\chi(G) = \lambda$ and any λ -coloring of G induces the same partition of $V(G)$ (vertices with the same color are in the same class). The chromatic polynomial $\chi(G, \lambda)$ gives, for all values λ , the number of proper λ -colorings of G . The famous 4-color Theorem states that, if G is a planar graph, then $\chi(G, 4) > 0$.

The connection with the Tutte polynomial is given by the following theorem.

Theorem 2.6. [11, Theorem 12 and 17] *Let $G = (V, E)$ be a graph. Then,*

$$(1) \ R(G, p) = p^{|V(G)|-1} (1-p)^{|E(G)|-|V(G)|+1} T\left(G; 1, \frac{1}{1-p}\right);$$

$$(2) \chi(G, \lambda) = (-1)^{r(G)} \lambda^{k(G)} T(G; 1 - \lambda, 0).$$

Finally, we want to recall the well-known connection between the Tutte polynomial of a graph and the Ising model on it, which is obtained as a special case of the Q -Potts model, for $Q = 2$.

The famous Ising model of ferromagnetism consists of discrete variables called spins arranged on the vertices of the graph G . Each spin can take values ± 1 and only interacts with its nearest neighbors. Configuration of spins at two adjacent vertices i and j has energy $J > 0$ if the spins have opposite values, and $-J$ if the values are the same. Let $|V(G)| = N$, and let $\vec{\sigma} = (\sigma_1, \dots, \sigma_N)$ denote the configuration of spins, with $\sigma_i \in \{\pm 1\}$. The total energy of the system in configuration $\vec{\sigma}$ is then

$$E(\vec{\sigma}) = -J \sum_{i \sim j} \sigma_i \sigma_j,$$

where $i \sim j$ means that the vertices i and j are adjacent in G . The probability of a particular configuration at temperature T is given by

$$\mathbb{P}(\vec{\sigma}) = \frac{1}{Z} \exp(-\beta E(\vec{\sigma})),$$

where β is the “inverse temperature” conventionally defined as $\beta \equiv 1/(k_B T)$, and k_B denotes the Boltzmann constant. As usual in statistical physics, the normalizing constant Z , that makes the distribution above a probability measure, is called the *partition function*:

$$Z = \sum_{\vec{\sigma}} \exp(-\beta E(\vec{\sigma})).$$

It is known [22] that the partition function Z of the Ising model on G can be obtained by evaluating the Tutte polynomial $T(G; x, y)$ on the hyperbola $(x-1)(y-1) = 2$. More precisely, one has:

$$Z = 2(e^{2\beta J} - 1)^{|V(G)|-1} e^{-\beta J|E(G)|} T\left(G; \frac{e^{2\beta J} + 1}{e^{2\beta J} - 1}, e^{2\beta J}\right).$$

In the next sections we will explicitly verify this correspondence for the Schreier graphs of both the Grigorchuk and Basilica groups, using the computations of the partition functions made in [9].

2.2. Groups of automorphisms of rooted regular trees. We recall some basic facts about self-similar groups. Let T_q be the infinite regular rooted tree of degree q , i.e., the rooted tree in which each vertex has q children. Each vertex of the n -th level of the tree can be regarded as a word of length n in the alphabet $X = \{0, 1, \dots, q-1\}$. Moreover, one can identify the set X^ω of infinite words in X with the set ∂T_q of infinite geodesic rays starting at the root of T_q . Next, let $S < \text{Aut}(T_q)$ be a group acting on T_q by automorphisms generated by a finite symmetric set of generators Y . Moreover, suppose that the action is transitive on each level of the tree.

Definition 2.7. *The n -th Schreier graph Σ_n of the action of S on T_q , with respect to the generating set Y , is a graph whose vertex set coincides with the set of vertices of the n -th level of the tree, and two vertices u, v are adjacent if and only if there exists $s \in Y$ such that $s(u) = v$. If this is the case, the edge joining u and v is labelled by s .*

The vertices of Σ_n are labelled by words of length n in X and the edges are labelled by elements of Y . The Schreier graph is thus a regular graph of degree $|Y|$ with q^n vertices, and it is connected, since the action of S is level-transitive.

Definition 2.8. [16] *A finitely generated group $S < \text{Aut}(T_q)$ is self-similar if, for all $g \in S, x \in X$, there exist $h \in S, y \in X$ such that*

$$g(xw) = yh(w),$$

for all finite words w in the alphabet X .

Self-similarity implies that S can be embedded into the wreath product $\text{Sym}(q) \wr S$, where $\text{Sym}(q)$ denotes the symmetric group on q elements, so that any automorphism $g \in S$ can be represented as

$$g = \tau(g_0, \dots, g_{q-1}),$$

where $\tau \in \text{Sym}(q)$ describes the action of g on the first level of T_q and $g_i \in S, i = 0, \dots, q-1$, is the restriction of g on the full subtree of T_q rooted at the vertex i of the first level of T_q (observe that any such subtree is isomorphic to T_q). Hence, if $x \in X$ and w is a finite word in X , we have

$$g(xw) = \tau(x)g_x(w).$$

In the next sections, the Schreier graphs of the Grigorchuk group and of the Basilica group will be described. For both of them, we recall some substitutional rules that allow to recursively construct these sequences of graphs, starting from the Schreier graph associated with the action of the group on the first level of the rooted binary tree.

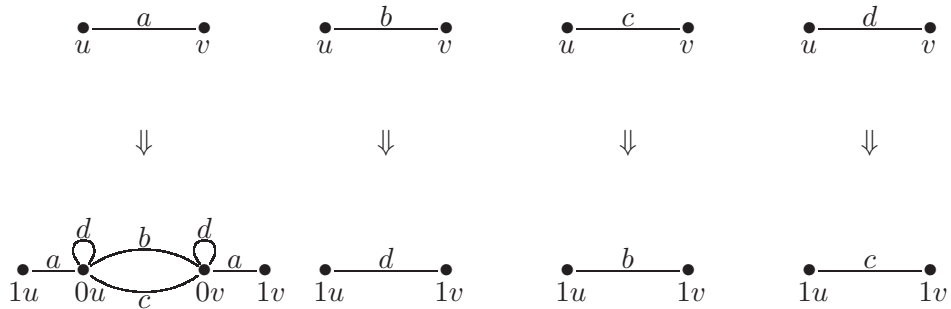
3. THE TUTTE POLYNOMIAL OF THE SCHREIER GRAPHS OF THE GRIGORCHUK GROUP

The Grigorchuk group admits the following description as a self-similar group of automorphisms of the rooted binary tree. It is generated by the elements

$$a = \epsilon(id, id), \quad b = e(a, c), \quad c = e(a, d), \quad d = e(id, b),$$

where e and ϵ are respectively the trivial and the non-trivial permutations in $\text{Sym}(2)$. Note that each generator is an involution.

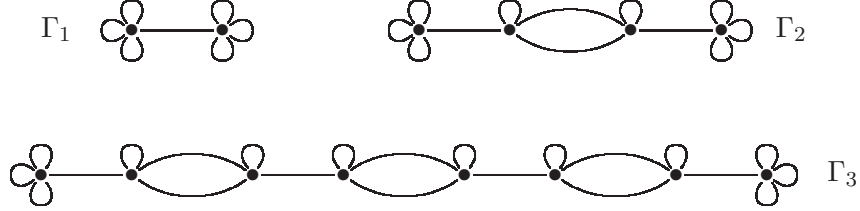
The following substitutional rules describe how to construct the graph Γ_{n+1} from Γ_n [1, 12]. More precisely, the construction consists in replacing the labelled subgraphs of Γ_n on the top of the picture by new labelled graphs (on the bottom).



The starting point is the Schreier graph Γ_1 of the first level.

$$\Gamma_1 \quad b, c, d \circlearrowleft \bullet_0 \xrightarrow{a} \bullet_1 \circlearrowright b, c, d$$

In computing the Tutte polynomial of Γ_n , we are interested in the unlabelled graph. We draw here the graphs Γ_n , for $n = 1, 2, 3$.



In general, one can check by using the substitutional rules that Γ_n has a linear shape, obtained by alternating bridges and 2-cycles. More precisely, it is easy to prove the following equalities:

$$|V(\Gamma_n)| = 2^n \quad |E(\Gamma_n)| = 5 \cdot 2^{n-1} + 2.$$

As regards edges, note that there are 2^{n-1} bridges, $2^{n-1} - 1$ cycles of length 2 and $2^n + 4$ loops, since there is a loop rooted at each vertex, except for the outmost vertices, where three loops are rooted.

Since many computations are trivial for graphs with loops, it is convenient to consider Γ_n^* , defined as the graph obtained from Γ_n by erasing loops. Thus, in this case, we have

$$|V(\Gamma_n^*)| = 2^n \quad |E(\Gamma_n^*)| = 3 \cdot 2^{n-1} - 2.$$

For every $n \geq 1$, denote by $T_n(x, y)$ the Tutte polynomial $T(\Gamma_n; x, y)$ of Γ_n and by $T_n^*(x, y)$ the Tutte polynomial $T(\Gamma_n^*; x, y)$ of Γ_n^* .

Theorem 3.1. *For each $n \geq 1$, the Tutte polynomial of the graph Γ_n is*

$$T_n(x, y) = y^{2^n+4} x^{2^{n-1}} (y + x)^{2^{n-1}-1}.$$

Proof. It suffices to apply the multiplicative property, keeping in mind that each loop, bridge or 2-cycle contributes by a factor y, x or $(y + x)$, respectively. \square

Corollary 3.2. *For each $n \geq 1$, the Tutte polynomial of the graph Γ_n^* is*

$$T_n^*(x, y) = x^{2^{n-1}} (y + x)^{2^{n-1}-1}.$$

Let us start by writing the reliability polynomial $R(\Gamma_n, p)$.

Proposition 3.3. *For each $n \geq 1$, the reliability polynomial $R(\Gamma_n, p)$ is given by*

$$R(\Gamma_n, p) = p^{2^n-1} (2 - p)^{2^{n-1}-1}.$$

Proof. It suffices to apply Equation (1) of Theorem 2.6. \square

Remark 3.4. Note that the existence of loops does not change the reliability polynomial; therefore $R(\Gamma_n, p) = R(\Gamma_n^*, p)$, as one can directly check.

As regards the complexity of Γ_n , the following proposition holds.

Proposition 3.5. *The complexity of Γ_n is $2^{2^{n-1}-1}$.*

Proof. According with Formula (1) of Theorem 2.5, it suffices to compute $T_n(1, 1)$. \square

Remark 3.6. The value of $\tau(\Gamma_n)$ has the following interpretation: each bridge of Γ_n must belong to any spanning subtree of Γ_n . On the other hand, a spanning subtree of Γ_n must contain exactly one edge of each 2-cycle. Then the result follows, since the number of 2-cycles of Γ_n is $2^{n-1} - 1$ and we have two choices for each 2-cycle. Also observe that loops do not contribute to $\tau(\Gamma_n)$; therefore, $T_n(1, 1) = T_n^*(1, 1)$ and so $\tau(\Gamma_n) = \tau(\Gamma_n^*)$, for each $n \geq 1$.

Corollary 3.7. *The asymptotic growth constant of the spanning trees of Γ_n is $\frac{1}{2} \log 2$.*

Proof. It suffices to compute

$$\lim_{n \rightarrow \infty} \frac{\log(\tau(\Gamma_n))}{|V(\Gamma_n)|},$$

with $|V(\Gamma_n)| = 2^n$. □

Evaluating $T_n(x, y)$ in $(1, 2)$ provides the number of connected spanning subgraphs of Γ_n . The following proposition holds.

Proposition 3.8. *The number of connected spanning subgraphs of Γ_n is $2^{2^n+4} \cdot 3^{2^{n-1}-1}$.*

Proof. It suffices to apply Formula (2) of Theorem 2.5. □

Remark 3.9. The value that we have found for the number of connected spanning subgraphs of Γ_n has the following interpretation: a connected spanning subgraph of Γ_n necessarily contains each bridge of the graph. On the other hand, both the edges or only one edge of each 2-cycle must belong to the subgraph (if no edge of a cycle belongs to the subgraph, then this subgraph is not connected), so that, for each of the $2^{n-1} - 1$ cycles of length 2, we have three possibilities. Finally, a connected spanning subgraph can also contain loops and so we have two possibilities for each of the $2^n + 4$ loops.

Another interesting computation concerns the number of spanning forests of Γ_n , which is given by $T_n(2, 1)$.

Proposition 3.10. *The number of spanning forests of Γ_n is $2^{2^{n-1}} \cdot 3^{2^{n-1}-1}$.*

Proof. It suffices to apply Formula (3) of Theorem 2.5. □

Remark 3.11. The value that we have found for the number of spanning forests of Γ_n has the following interpretation: a spanning forest of Γ_n cannot contain loops nor both the edges of a 2-cycle, since this would produce a cycle. Therefore, no edges or only one edge of each 2-cycle must belong to the forest. On the other hand, each bridge can belong to a spanning forest of Γ_n . Since the number of 2-cycles is $2^{n-1} - 1$ and the number of bridges is 2^{n-1} , we get the result.

Next, we explicitly verify that by evaluating the Tutte polynomial of Γ_n in $(2, 2)$ one gets $2^{|E(\Gamma_n)|}$ (see Formula (4) of Theorem 2.5).

Proposition 3.12. *For each $n \geq 1$, one has $T_n(2, 2) = 2^{|E(\Gamma_n)|} = 2^{5 \cdot 2^{n-1} + 2}$.*

Proof. By definition of $T_n(x, y)$, one has:

$$T_n(2, 2) = 2^{2^n+4} \cdot 2^{2^{n-1}} \cdot 4^{2^{n-1}-1} = 2^{5 \cdot 2^{n-1} + 2}.$$

□

Finally, by evaluating the Tutte polynomial of Γ_n in $(2, 0)$, we investigate the number of acyclic orientations of Γ_n . Observe that, whenever we have loops, the number of possible acyclic orientations on the graphs is 0. Therefore, we consider the graphs $\{\Gamma_n^*\}_{n \geq 1}$ without loops, whose Tutte polynomial is $T_n^*(x, y) = x^{2^{n-1}}(x + y)^{2^{n-1}-1}$.

Proposition 3.13. *The number of acyclic orientations on Γ_n^* is 2^{2^n-1} .*

Proof. By definition of $T_n^*(x, y)$, one has:

$$T_n^*(2, 0) = 2^{2^{n-1}} \cdot 2^{2^{n-1}-1} = 2^{2^n-1}.$$

□

Remark 3.14. The value that we have found for the number of acyclic orientations of Γ_n^* has the following interpretation: we have two possible orientations for each bridge, giving the factor $2^{2^{n-1}}$. Then, each 2-cycle can receive four orientations, as shown in the following picture.



Only the first two orientations are acyclic, and so the 2-cycles give a contribution equal to $2^{2^{n-1}-1}$.

Since Γ_n has loops, it does not admit any proper coloring, so that we investigate the chromatic polynomial of Γ_n^* .

Proposition 3.15. *For each $n \geq 1$, the chromatic polynomial $\chi_n(\lambda)$ of Γ_n^* is*

$$\chi_n(\lambda) = -\lambda(1 - \lambda)^{2^n-1}.$$

Proof. By applying Equation (2) of Theorem 2.6, one gets

$$\chi_n(\lambda) = (-1)^{2^n-1} \lambda(1 - \lambda)^{2^n-1} (1 - \lambda)^{2^{n-1}-1} = -\lambda(1 - \lambda)^{2^n-1}.$$

□

Remark 3.16. Note that $\chi_n(2) = 2$, for each $n \geq 1$, according to the fact that the graph is bipartite and so uniquely 2-colorable.

We end this section by investigating the relationship between the evaluation of the Tutte polynomial of the Schreier graph Γ_n^* on the hyperbola $(x - 1)(y - 1) = 2$ and the partition function of the Ising model on the same graph. In [9, Theorem 2.1], the partition function of the Ising model on Γ_n^* has been described as

$$Z_n = 2^{2^n} \cosh(\beta J)^{3 \cdot 2^{n-1}-2} (1 + \tanh^2(\beta J))^{2^{n-1}-1}.$$

Theorem 3.17. *For each $n \geq 1$, one has*

$$(3) \quad 2(e^{2\beta J} - 1)^{|V(\Gamma_n^*)|-1} \cdot e^{-\beta J|E(\Gamma_n^*)|} \cdot T_n^*\left(\frac{e^{2\beta J} + 1}{e^{2\beta J} - 1}, e^{2\beta J}\right) = Z_n.$$

Proof. Recall that $|E(\Gamma_n^*)| = 3 \cdot 2^{n-1} - 2$ and $|V(\Gamma_n^*)| = 2^n$. Let $e^{\beta J} = t$, so that Equation (3) can be written as

$$(4) \quad \frac{2(t^2 - 1)^{2^{n-1}}}{t^{3 \cdot 2^{n-1}-2}} \cdot T_n^*\left(\frac{t^2 + 1}{t^2 - 1}, t^2\right) = 2^{2^n} \left(\frac{t^2 + 1}{2t}\right)^{3 \cdot 2^{n-1}-2} \left(1 + \left(\frac{t^2 - 1}{t^2 + 1}\right)^2\right)^{2^{n-1}-1}.$$

One can directly check that

$$T_n^* \left(\frac{t^2 + 1}{t^2 - 1}, t^2 \right) = \frac{(t^2 + 1)^{2^{n-1}} (t^4 + 1)^{2^{n-1}-1}}{(t^2 - 1)^{2^{n-1}}}.$$

Then, it is not difficult to prove that both sides of Equation (4) are equal to

$$\frac{2(t^2 + 1)^{2^{n-1}} (t^4 + 1)^{2^{n-1}-1}}{t^{3 \cdot 2^{n-1}-2}}.$$

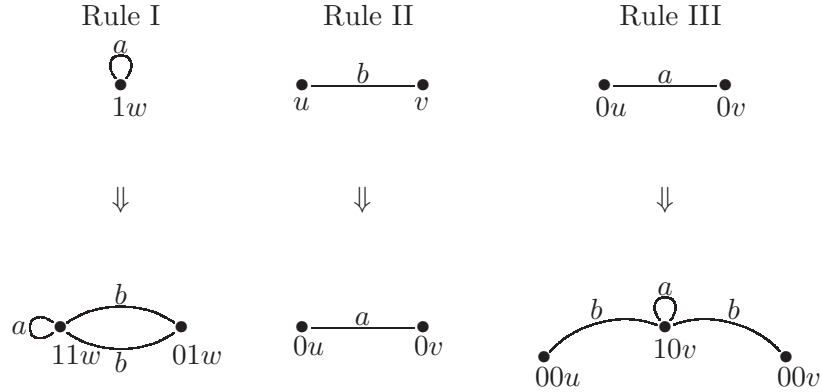
□

4. THE TUTTE POLYNOMIAL OF THE SCHREIER GRAPHS OF THE BASILICA GROUP

The Basilica group is a self-similar group of automorphisms of the rooted binary tree generated by the elements

$$a = e(b, id), \quad b = \epsilon(a, id).$$

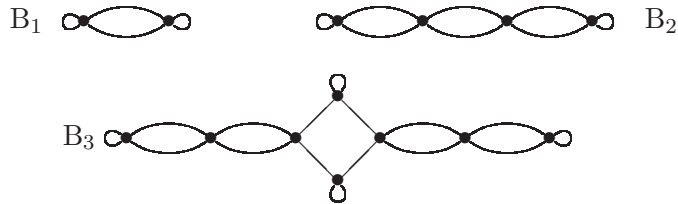
The associated Schreier graphs can be recursively constructed via the following substitutional rules:

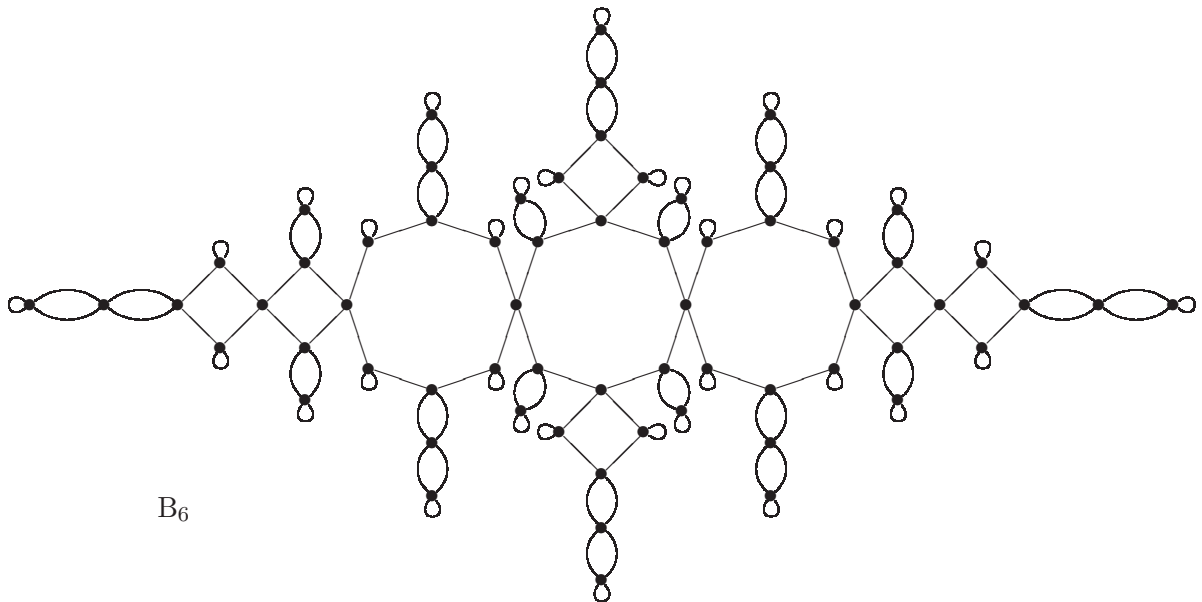
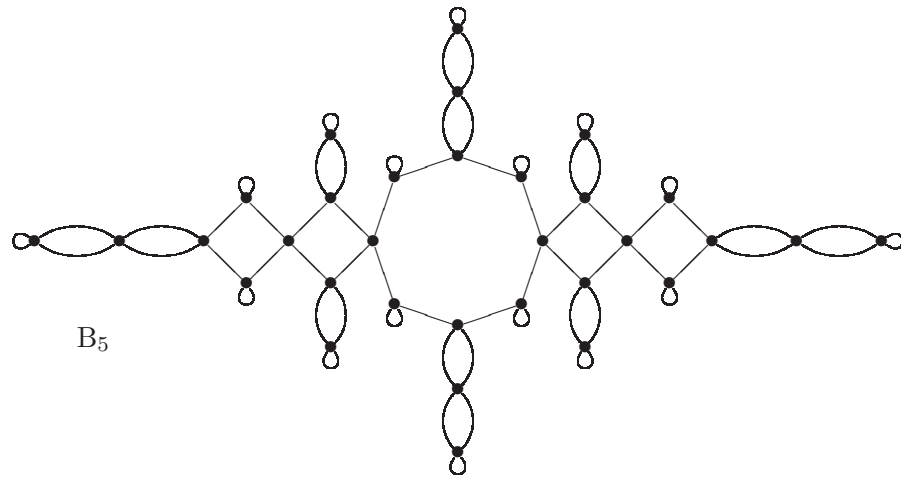
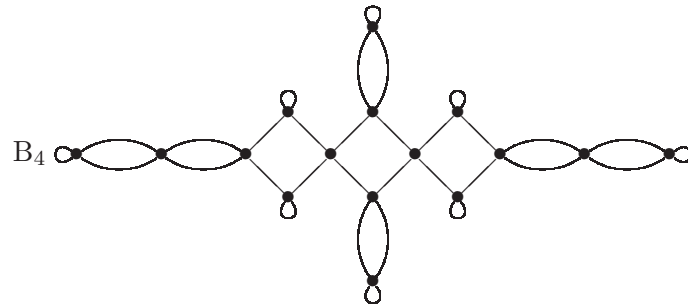


The starting point is the Schreier graph B_1 of the first level:



As in the case of the Grigorchuk group, we are interested in the unlabelled Schreier graphs. The following pictures of graphs B_n for $n = 1, 2, 3, 4, 5, 6$ give an idea of how Schreier graphs of the Basilica group look like. See [8] for a comprehensive analysis of finite and infinite Schreier graphs of this group. Note also that $\{B_n\}_{n \geq 1}$ is an approximating sequence for the Julia set of the polynomial $z^2 - 1$, the famous “Basilica” fractal (see [16]).





In general, it follows from the recursive definition of the generators, that each B_n is a cactus, i.e., a union of cycles (all of them are of length power of 2) arranged in a tree-like way. The maximal length of a cycle in B_n is $2^{\frac{n+1}{2}}$ if n is odd and $2^{\frac{n}{2}}$ if n is even. Moreover, for each $n \geq 2$, the graph B_n contains exactly 2^{n-1} loops rooted at the vertices corresponding to words in the alphabet $\{0, 1\}$ starting by 1, since the action of the generator a on these words is trivial.

Proposition 4.1. *For any $n \geq 4$, the number $c_{n,i}$ of cycles of length 2^i in B_n is:*

$$c_{n,i} = \begin{cases} 3 \cdot 2^{n-2i-1} & \text{for } 1 \leq i \leq \frac{n}{2} - 1, \\ 3 & \text{for } i = \frac{n}{2} \end{cases}, \quad \text{for } n \text{ even,}$$

and

$$c_{n,i} = \begin{cases} 3 \cdot 2^{n-2i-1} & \text{for } 1 \leq i \leq \frac{n-1}{2} - 1 \\ 4 & \text{for } i = \frac{n-1}{2} \\ 1 & \text{for } i = \frac{n+1}{2} \end{cases}, \quad \text{for } n \text{ odd.}$$

Proof. It follows from [9, Proposition 2.2]. □

To sum up, we have

$$|V(B_n)| = 2^n \quad |E(B_n)| = 2^{n+1}.$$

As regards edges, note that there are 2^{n-1} loops, for $n \geq 2$, and 2 loops in B_1 .

Since many computation are trivial for graphs with loops, it is convenient to define B_n^* as the graph B_n considered without loops. Thus, in this case, we have

$$|V(B_n^*)| = 2^n \quad |E(B_n^*)| = 3 \cdot 2^{n-1}.$$

For every $n \geq 1$, denote by $T_n(x, y)$ the Tutte polynomial $T(B_n; x, y)$ of B_n and by $T_n^*(x, y)$ the Tutte polynomial $T(B_n^*; x, y)$ of B_n^* .

Theorem 4.2. *For $n \geq 4$, the Tutte polynomial of the Schreier graph B_n of the Basilica group is*

$$(5) \quad T_n(x, y) = y^{2^{n-1}} (y + x + \cdots + x^{2^{\frac{n-1}{2}-1}})^4 (y + x + \cdots + x^{2^{\frac{n+1}{2}-1}}) \cdot \prod_{i=1}^{\frac{n-1}{2}-1} (y + x + \cdots + x^{2^i-1})^{3 \cdot 2^{n-2i-1}}$$

for n odd and

$$(6) \quad T_n(x, y) = y^{2^{n-1}} (y + x + \cdots + x^{2^{\frac{n}{2}-1}})^3 \prod_{i=1}^{\frac{n}{2}-1} (y + x + \cdots + x^{2^i-1})^{3 \cdot 2^{n-2i-1}}$$

for n even. Moreover, one has

$$T_1(x, y) = y^2(x + y) \quad T_2(x, y) = y^2(x + y)^3 \quad T_3(x, y) = y^4(x + y)^4(y + x + x^2 + x^3).$$

Proof. The proof follows from Proposition 4.1 and Lemma 2.4. More precisely, by multiplicative property (2), $T_n(x, y)$ is obtained as the product of the Tutte polynomials of its cycles. □

Corollary 4.3. *For each $n \geq 4$, the Tutte polynomial of the graph B_n^* is*

$$T_n^*(x, y) = (y + x + \cdots + x^{2^{\frac{n-1}{2}}-1})^4 (y + x + \cdots + x^{2^{\frac{n+1}{2}}-1}) \prod_{i=1}^{\frac{n-1}{2}-1} (y + x + \cdots + x^{2^i-1})^{3 \cdot 2^{n-2i-1}}$$

for n odd and

$$T_n^*(x, y) = (y + x + \cdots + x^{2^{\frac{n}{2}}-1})^3 \prod_{i=1}^{\frac{n}{2}-1} (y + x + \cdots + x^{2^i-1})^{3 \cdot 2^{n-2i-1}}$$

for n even. Moreover, one has

$$T_1^*(x, y) = (x + y) \quad T_2^*(x, y) = (x + y)^3 \quad T_3^*(x, y) = (x + y)^4 (y + x + x^2 + x^3).$$

Let us start by writing the reliability polynomial $R(B_n, p)$.

Proposition 4.4. *For each $n \geq 4$, the reliability polynomial $R(B_n, p)$ is given by*

$$R(B_n, p) = p^{2^n-1} (1-p)^{2^{n-1}+1} \left(2^{\frac{n-1}{2}} + \frac{p}{1-p}\right)^4 \left(2^{\frac{n+1}{2}} + \frac{p}{1-p}\right) \prod_{i=1}^{\frac{n-1}{2}-1} \left(\frac{p}{1-p} + 2^i\right)^{3 \cdot 2^{n-2i-1}}$$

for n odd and

$$R(B_n, p) = p^{2^n-1} (1-p)^{2^{n-1}+1} \left(2^{\frac{n}{2}} + \frac{p}{1-p}\right)^3 \prod_{i=1}^{\frac{n}{2}-1} \left(\frac{p}{1-p} + 2^i\right)^{3 \cdot 2^{n-2i-1}}$$

for n even. Moreover, one has

$$R(B_1, p) = p(2-p) \quad R(B_2, p) = p^3(2-p)^3 \quad R(B_3, p) = p^7(2-p)^4(4-3p).$$

Proof. It suffices to apply Equation (1) of Theorem 2.6. □

Remark 4.5. Note that the existence of loops does not change the reliability polynomial; therefore $R(B_n, p) = R(B_n^*, p)$, as one can directly check.

Evaluating $T_n(x, y)$ in $(1, 1)$, we get the complexity $\tau(B_n)$, i.e., the number of spanning trees of B_n .

Proposition 4.6. *The complexity of B_n is*

$$\tau(B_n) = \begin{cases} 2^{\frac{2^{n+2}+3n-5}{6}} & \text{for } n \text{ odd} \\ 2^{\frac{2^{n+2}+3n-4}{6}} & \text{for } n \text{ even} \end{cases}.$$

Proof. It suffices to apply Formula (1) of Theorem 2.5. Indeed, one gets

$$T_n(1, 1) = 2^{\frac{4(n-1)}{2}} \cdot 2^{\frac{n+1}{2}} \prod_{i=1}^{\frac{n-1}{2}-1} 2^{i \cdot 3 \cdot 2^{n-2i-1}} = 2^{\frac{2^{n+2}+3n-5}{6}}$$

for n odd and

$$T_n(1, 1) = 2^{\frac{3n}{2}} \prod_{i=1}^{\frac{n}{2}-1} 2^{i \cdot 3 \cdot 2^{n-2i-1}} = 2^{\frac{2^{n+2}+3n-4}{6}}$$

for n even. For $n = 1, 2, 3$, one can directly find $T_1(1, 1) = 2$, $T_2(1, 1) = 2^3$, $T_3(1, 1) = 2^6$. □

Remark 4.7. The previous equations can be motivated in the following way. In order to have a spanning tree of B_n , we do not have to consider loops; moreover, we have to delete exactly one edge in every cycle, so that every cycle of length 2^i contributes by a factor 2^i . Since the loops do not contribute to the number of spanning trees of B_n , note that $\tau(B_n) = \tau(B_n^*)$, for each $n \geq 1$.

Corollary 4.8. *The asymptotic growth constant of the spanning trees of B_n is $\frac{2}{3} \log 2$.*

Proof. It suffices to compute

$$\lim_{n \rightarrow \infty} \frac{\log(\tau(B_n))}{|V(B_n)|},$$

with $|V(B_n)| = 2^n$. □

The evaluation of $T_n(x, y)$ in (1, 2) provides the number of connected spanning subgraphs of B_n . More precisely, the following proposition holds.

Proposition 4.9. *The number of connected spanning subgraphs of B_n is*

$$T_n(1, 2) = 2^{2^{n-1}} \left(1 + 2^{\frac{n-1}{2}}\right)^4 \left(1 + 2^{\frac{n+1}{2}}\right) \prod_{i=1}^{\frac{n-1}{2}-1} (1 + 2^i)^{3 \cdot 2^{n-2i-1}}$$

for n odd and

$$T_n(1, 2) = 2^{2^{n-1}} \left(1 + 2^{\frac{n}{2}}\right)^3 \prod_{i=1}^{\frac{n}{2}-1} (1 + 2^i)^{3 \cdot 2^{n-2i-1}}$$

for n even. Moreover, one has

$$T_1(1, 2) = 2^2 \cdot 3 \quad T_2(1, 2) = 2^2 \cdot 3^3 \quad T_3(1, 2) = 2^4 \cdot 3^4 \cdot 5.$$

Proof. It suffices to apply Formula (2) of Theorem 2.5. □

Remark 4.10. The value that we have found for the number of connected spanning subgraphs of B_n has the following interpretation: the factor $2^{2^{n-1}}$ corresponds to the possibility of choosing loops in the subgraph. On the other hand, in order to get a connected spanning subgraph, each cycle of length 2^i contributes by a factor $2^i + 1$, since we can take the whole cycle or delete exactly one edge from it.

Another interesting computation concerns the number of spanning forests of the Schreier graph B_n , which is given by $T_n(2, 1)$.

Proposition 4.11. *The number of spanning forests of B_n is*

$$T_n(2, 1) = \left(2^{2^{\frac{n-1}{2}}} - 1\right)^4 \left(2^{2^{\frac{n+1}{2}}} - 1\right) \prod_{i=1}^{\frac{n-1}{2}-1} (2^{2^i} - 1)^{3 \cdot 2^{n-2i-1}}$$

for n odd and

$$T_n(2, 1) = \left(2^{2^{\frac{n}{2}}} - 1\right)^3 \prod_{i=1}^{\frac{n}{2}-1} (2^{2^i} - 1)^{3 \cdot 2^{n-2i-1}}$$

for n even. Moreover, one has

$$T_1(2, 1) = 3 \quad T_2(2, 1) = 3^3 \quad T_3(2, 1) = 3^5 \cdot 5.$$

Proof. It suffices to apply Formula (3) of Theorem 2.5. \square

Remark 4.12. The value that we have found for the number of spanning forests of B_n has the following interpretation: a spanning forest of B_n cannot contain loops nor the whole cycles, since this would produce a cycle. Therefore, we can choose or not each edge of any 2^i -cycle, but we cannot choose in a spanning forest all the edges of the cycle. Hence, a 2^i -cycle contributes to the number of spanning forests by a factor $2^{2^i} - 1$. Moreover, observe that $T_n(2, 1) = T_n^*(2, 1)$, for each $n \geq 1$.

Next, we explicitly verify that by evaluating the Tutte polynomial of B_n in $(2, 2)$ one gets $2^{|E(B_n)|}$ (see Formula (4) of Theorem 2.5).

Proposition 4.13. *For each $n \geq 1$, one has $T_n(2, 2) = 2^{|E(B_n)|} = 2^{2^{n+1}}$.*

Proof. By replacing $x = y = 2$ in Equations (5) and (6), it turns out that each cycle of length 2^i contributes by a factor 2^{2^i} to $T_n(2, 2)$. \square

Finally, by evaluating the Tutte polynomial of B_n in $(2, 0)$, we investigate the number of acyclic orientations of B_n . Observe that, whenever we have loops, the number of possible acyclic orientations on the graphs is 0. Therefore, we consider the graph B_n^* .

Proposition 4.14. *The number of acyclic orientations on B_n^* is*

$$T_n^*(2, 0) = \left(2^{2^{\frac{n-1}{2}}} - 2\right)^4 \left(2^{2^{\frac{n+1}{2}}} - 2\right) \prod_{i=1}^{\frac{n-1}{2}-1} \left(2^{2^i} - 2\right)^{3 \cdot 2^{n-2i-1}}$$

for n odd and

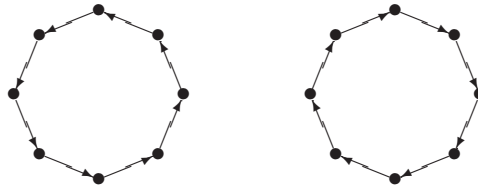
$$T_n^*(2, 0) = \left(2^{2^{\frac{n}{2}}} - 2\right)^3 \prod_{i=1}^{\frac{n}{2}-1} \left(2^{2^i} - 2\right)^{3 \cdot 2^{n-2i-1}}$$

for n even. Moreover, one has

$$T_1^*(2, 0) = 2 \quad T_2^*(2, 0) = 2^3 \quad T_3^*(2, 0) = 2^5 \cdot 7.$$

Proof. By replacing $x = 2$ and $y = 0$ in Equations (5) and (6), it turns out that each cycle of length 2^i contributes by a factor $2^{2^i} - 2$ to $T_n^*(2, 0)$. \square

Remark 4.15. The fact that each cycle of length 2^i contributes by the factor $2^{2^i} - 2$ has the following interpretation: there are two possible orientations for each edge, but we have to avoid the two cases where we create an oriented cycle (represented in the following picture in the case of a 2^3 -cycle).



Since B_n has loops, it does not admit any proper coloring, so that we investigate the chromatic polynomial of B_n^* .

Proposition 4.16. *For each $n \geq 4$, the chromatic polynomial $\chi_n(\lambda)$ of B_n^* is*

$$\chi_n(\lambda) = -\lambda \left(\frac{1-\lambda}{\lambda} - \frac{(1-\lambda)^{2^{\frac{n-1}{2}}}}{\lambda} \right)^4 \left(\frac{1-\lambda}{\lambda} - \frac{(1-\lambda)^{2^{\frac{n+1}{2}}}}{\lambda} \right)^{\frac{n-1}{2}-1} \prod_{i=1}^{\frac{n-1}{2}-1} \left(\frac{1-\lambda}{\lambda} - \frac{(1-\lambda)^{2^i}}{\lambda} \right)^{3 \cdot 2^{n-2i-1}}$$

for n odd and

$$\chi_n(\lambda) = -\lambda \left(\frac{1-\lambda}{\lambda} - \frac{(1-\lambda)^{2^{\frac{n}{2}}}}{\lambda} \right)^3 \prod_{i=1}^{\frac{n}{2}-1} \left(\frac{1-\lambda}{\lambda} - \frac{(1-\lambda)^{2^i}}{\lambda} \right)^{3 \cdot 2^{n-2i-1}}$$

for n even. Moreover, one has

$$\chi_1(\lambda) = -\lambda(1-\lambda) \quad \chi_2(\lambda) = -\lambda(1-\lambda)^3 \quad \chi_3(\lambda) = -\lambda(1-\lambda)^5 \cdot (\lambda^2 - 3\lambda + 3).$$

Proof. It suffices to apply Equation (2) of Theorem 2.6. \square

Remark 4.17. Note that $\chi_n(2) = 2$, for each $n \geq 1$, according to the fact that the graph is bipartite and so uniquely 2-colorable.

Finally, we investigate the relationship between the evaluation of the Tutte polynomial of the Schreier graph B_n^* on the hyperbola $(x-1)(y-1) = 2$ and the partition function of the Ising model on the same graph. In [9, Theorem 2.4], the partition function of the Ising model on B_n^* has been described as

$$Z_n = 2^{2^n} (\cosh(\beta J))^{3 \cdot 2^{n-1}} \Phi_n(\tanh(\beta J)),$$

where $\Phi_n(z)$ is the generating function of closed polygons for B_n^* given by

$$\Phi_n(z) = \left(1 + z^{2^{\frac{n-1}{2}}} \right)^4 \left(1 + z^{2^{\frac{n+1}{2}}} \right)^{\frac{n-1}{2}-1} \prod_{i=1}^{\frac{n-1}{2}-1} \left(1 + z^{2^i} \right)^{3 \cdot 2^{n-2i-1}}$$

for $n \geq 5$ odd and

$$\Phi_n(z) = \left(1 + z^{2^{\frac{n}{2}}} \right)^3 \prod_{i=1}^{\frac{n}{2}-1} \left(1 + z^{2^i} \right)^{3 \cdot 2^{n-2i-1}}$$

for $n \geq 4$ even. Moreover,

$$Z_1 = 2^2 \cosh^2(\beta J) (1 + \tanh^2(\beta J))$$

$$Z_2 = 2^4 \cosh^6(\beta J) (1 + \tanh^2(\beta J))^3$$

$$Z_3 = 2^8 \cosh^{12}(\beta J) (1 + \tanh^2(\beta J))^4 (1 + \tanh^4(\beta J)).$$

Theorem 4.18. *For each $n \geq 1$, one has*

$$(7) \quad 2(e^{2\beta J} - 1)^{|V(B_n^*)|-1} \cdot e^{-\beta J|E(B_n^*)|} \cdot T_n^* \left(\frac{e^{2\beta J} + 1}{e^{2\beta J} - 1}, e^{2\beta J} \right) = Z_n.$$

Proof. Here we only prove the case of n even (the computations for n odd are similar). Recall that $|E(B_n^*)| = 3 \cdot 2^{n-1}$ and $|V(B_n^*)| = 2^n$. Let $e^{\beta J} = t$, so that Equation (7) can be written as

$$(8) \quad \frac{2(t^2-1)^{2^{n-1}}}{t^{3 \cdot 2^{n-1}}} \cdot T_n^* \left(\frac{t^2+1}{t^2-1}, t^2 \right) = 2^{2^n} \left(\frac{t^2+1}{2t} \right)^{3 \cdot 2^{n-1}} \left(1 + z^{2^{\frac{n}{2}}} \right)^3 \prod_{i=1}^{\frac{n}{2}-1} \left(1 + z^{2^i} \right)^{3 \cdot 2^{n-2i-1}} \Big|_{z=\frac{t^2-1}{t^2+1}}.$$

One can directly check that

$$T_n^* \left(\frac{t^2+1}{t^2-1}, t^2 \right) = \left(\frac{t^2-1}{2} \left(1 + \left(\frac{t^2+1}{t^2-1} \right)^{2^{\frac{n}{2}}} \right) \right)^{3 \cdot \frac{n}{2} - 1} \prod_{i=1}^{3 \cdot \frac{n}{2} - 1} \left(\frac{t^2-1}{2} \left(1 + \left(\frac{t^2+1}{t^2-1} \right)^{2^i} \right) \right)^{3 \cdot 2^{n-2i-1}}$$

Then, it is not difficult to prove that both sides of Equation (4) are equal to

$$\frac{(t^2+1)^{3 \cdot 2^{n-1}}}{t^{3 \cdot 2^{n-1}} 2^{2^{n-1}}} \left(1 + \left(\frac{t^2-1}{t^2+1} \right)^{2^{\frac{n}{2}}} \right)^{3 \cdot \frac{n}{2} - 1} \prod_{i=1}^{3 \cdot \frac{n}{2} - 1} \left(1 + \left(\frac{t^2-1}{t^2+1} \right)^{2^i} \right)^{3 \cdot 2^{n-2i-1}}.$$

□

REFERENCES

- [1] L. Bartholdi and R. Grigorchuk, On the spectrum of Hecke type operators related to some fractal groups, *Tr. Mat. Inst. Steklova* **231** (2000), Din. Sist., Avtom. i Beskon. Gruppy, 5–45; translation in *Proc. Steklov Inst. Math.* (2000), no. 4 (**231**), 1–41.
- [2] L. Bartholdi, R. Grigorchuk and V. Nekrashevych, From fractal groups to fractal sets, in: “Fractals in Graz” (P. Grabner and W. Woess editors), Trends in Mathematics, Birkhäuser Verlag, Basel, 2003, 25–118.
- [3] L. Bartholdi and B. Virág, Amenability via random walks. *Duke Math Journal*. **130** (2005), no. 1, 39–56.
- [4] B. Bollobás, *Modern Graph Theory*, Graduate Texts in Mathematics, **184**. Springer-Verlag, New York (1998).
- [5] T. Brylawski and J. Oxley, The Tutte polynomial and its Applications, in: “Matroid Applications” (123–225), (N. White editor), *Encyclopedia Math. Appl.*, **40**, Cambridge Univ. Press, Cambridge, 1992.
- [6] T. Ceccherini-Silberstein, A. Machì and F. Scarabotti, Il gruppo di Grigorchuk di crescita intermedia. *Rend. Circ. Mat. Palermo*. **50** (2001), no. 1, 67–102.
- [7] R. Cori and Y. Le Borgne, The sand-pile model and Tutte polynomials. Formal power series and algebraic combinatorics (Scottsdale, AZ, 2001), *Adv. in Appl. Math.*, **30**, Issue 1–2, (2003), 44–52.
- [8] D. D’Angeli, A. Donno, M. Matter and T. Nagnibeda, Schreier graphs of the Basilica group, *Journal of Modern Dynamics*, **4**, no. 1, (2010), 167–205.
- [9] D. D’Angeli, A. Donno and T. Nagnibeda, Partition functions of the Ising model on some self-similar Schreier graphs, to appear in the Proceedings of the Euroconference “Boundaries”, Graz, June 2009, (W. Woess and F. Sobieczky editors), Birkhäuser, 2010; available at <http://arxiv.org/abs/1003.0611>
- [10] A. Donno and D. Iacono, The Tutte polynomial of some self-similar graphs, submitted; available at <http://arxiv.org/abs/1006.5333>.
- [11] J. A. Ellis-Monaghan and C. Merino, *Graph Polynomials and Their Applications I: The Tutte Polynomial*, in: “Structural Analysis of Complex Networks” (Mathias Dehmer editor), in press; available at <http://arxiv.org/abs/0803.3079v2>
- [12] R. Grigorchuk, Solved and unsolved problems around one group, Infinite groups: geometric, combinatorial and dynamical aspects, 117–218, *Progr. Math.*, **248**, Birkhäuser, Basel, 2005.
- [13] R. Grigorchuk and A. Żuk, On a torsion-free weakly branch group defined by a three-state automaton. *International J. Algebra Comput.* **12** (2002), no. 1, 223–246.
- [14] R. Lyons, Asymptotic enumeration of spanning trees, *Combin. Probab. Comput.*, **14**, No. 4, (2005), 491–522.
- [15] C. Merino, Chip-firing and the Tutte polynomial, *Ann. Comb.*, **1**, No. 3, (1997), 253–259.
- [16] V. Nekrashevych, *Self-similar Groups*, Mathematical Surveys and Monographs, **117**. American Mathematical Society, Providence, RI, 2005.
- [17] J.G. Oxley and D.J.A. Welsh, The Tutte polynomial and percolation, in: “Graph theory and related topics” (Proc. Conf., Univ. Waterloo, Waterloo, Ont., 1977), 329–339, (J.A. Bondy and U.S.R. Murty editors), Academic Press, New York-London, 1979.
- [18] W.T. Tutte, A ring in graph theory, *Proc. Cambridge Philos. Soc.*, **43**, (1947), 26–40.
- [19] W.T. Tutte, A contribution to the theory of chromatic polynomials, *Canadian. J. Math.*, **6**, (1954), 80–91.
- [20] W.T. Tutte, On dichromatic polynomials, *J. Combinatorial Theory*, **2**, (1967), 301–320.
- [21] W.T. Tutte, Graph-polynomials. Special issue on the Tutte polynomial, *Adv. in Appl. Math.*, **32**, No. 1–2, (2004), 5–9.
- [22] D.J.A. Welsh and C. Merino, The Potts model and the Tutte polynomial. Probabilistic techniques in equilibrium and nonequilibrium statistical physics, *J. Math. Phys.*, **41**, No. 3, (2000), 1127–1152.

TULLIO CECCHERINI-SILBERSTEIN, DIPARTIMENTO DI INGEGNERIA, UNIVERSITÀ DEL SANNIO, C.SO GARIBALDI, 107, 82100 BENEVENTO, ITALIA

E-mail address: `tceccher@mat.uniroma1.it`

ALFREDO DONNO, DIPARTIMENTO DI MATEMATICA “G. CASTELNUOVO”, SAPIENZA UNIVERSITÀ DI ROMA, PIAZZALE A. MORO, 2, 00185 ROMA, ITALIA

E-mail address: `donno@mat.uniroma1.it`

DONATELLA IACONO, DIPARTIMENTO DI MATEMATICA “G. CASTELNUOVO”, SAPIENZA UNIVERSITÀ DI ROMA, PIAZZALE A. MORO, 2, 00185 ROMA, ITALIA

E-mail address: `iacono@mat.uniroma1.it`